Analysis of 3D Pendulum sliding along a Rope

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ABSTRACT

The paper deals with a dynamic engineering problem in which a mass attached to a pendulum slides along a cable. In this problem, the pendulum mass and the cable are coupled in a model described by a system of algebraic differential equations (DAE). In the paper, the formulation of the system of differential equations that models the problem is presented along with the determination of the initial conditions. The developed model is general in the sense of free choice of the support location, elastic rope properties, pendulum length and inclusion of braking forces. This model is applicable in the design of real rope structures, such as ropeways (zip-lines). Many calculated values that can be measured on a real structure can be exported from the model and used for parameter calibration. In an example, the model is related to a real cable structure for illustration and validation of the model.

Keywords: Elastic 3D rope/cable, Sliding mass, Sliding 3D pendulum, Differential algebraic equations, rope/cable measurements.

1 INTRODUCTION

The article deals with a dynamic engineering problem in which a mass attached to a pendulum slides along an elastic rope. This type of problem describes many practical applications such as weight manipulation, construction site transportation, cableways, etc. Usually, the centre of gravity of the mass is outside the rope axis, which is modelled with a pendulum attached to the rope. A similar problem was analysed in [1] and this is the 3D extension of this work. The terms 'cable' and 'rope' are used interchangeably throughout the paper to emphasise that the cable is massless in the model.

The pendulum mass and the cable are coupled in a model described by a system of algebraic differential equations (DAE). The initial conditions for the given system of DAE are formulated and solved separately. From the literature, finite elements are the preferred method for solving structural problems, but here the moving mass is a part of the structure, which requires the development of special finite elements [2]. However, searching the available sources, no finite element formulation was found that would account for the mass suspended from a pendulum and realistically describe the engineering problem of a sliding body. No analytical formulation of the problem can be found in the literature. Note that sliding along the wire should be distinguished from sliding of the wire itself [3]. In the sliding mass problem, there is no equilibrium without the sliding mass, i.e., the wire and the mass are coupled to form a system (in fact, the wire imposes nonlinear constraints on the dynamic equations of mass motion).

We have developed a novel analytical model for a mass attached to a pendulum sliding along a three-dimensional massless rope or cable. Since we want to solve a real engineering problem, it is important to take into account the elastic properties of the cable (elongation of the cable under tensile load). The developed model is general in the sense of free choice of the support point, elastic rope properties, pendulum length and inclusion of braking forces (or friction coefficient).

Further generalisation would include multiple successive masses [4].

An example is provided at the end of the paper to verify and illustrate the model. Some of the model results are presented graphically. They can be compared with field measurements and used for parameter calibration and rope/cable design improvement.

The novelty of the model is that it successfully describes a mass suspended from a 3D pendulum sliding along a 3D cable. This approach combines the two independent formulations: i) the sliding of a mass along an extensible 3D rope and ii) the motion of a 3D pendulum in the general translational coordinate system. Moreover, this analytical approach can be used to test different discrete formulations of the problem.

2 SLIDING PENDULUM MODEL

The rope imposes nonlinear constraints on the dynamic equations of a point mass sliding along the rope and creates a translational origin for the mass swinging on the pendulum. The two systems of equations are combined into a system of 10 normalised, i.e., first-order algebraic differential equations.

See Table 1 for the meaning of the variables in Figures 1 and 2.

Table 1. *Description of the variables used in the model*

a - position on the rope	u - longitudinal velocity of rope at 'a'
F - displacement of the rope at 'a'	v - vertical velocity of rope at 'a'
Y - lateral displacement of the rope at 'a'	w - lateral velocity of rope at 'a'
ϑ - pendulum angle along to the rope	p - velocity (speed of change) of ϑ
ψ - pendulum angle perpendicular to the rope	q - velocity (speed of change) of ψ
EA - cable elastic properties	L - cable length
m - mass of the sliding body	l - projected cable length
T - tension force in the cable	h - height between cable supports

2.1 The sliding mass model

The sliding mass model assumes a straight rope, i.e., the dead weight of the rope is neglected and the rope sag is rather small. This model is suitable for the analysis of light ropes and thin steel cables with very low sag. The description of the dynamic equilibrium of mass leads to three second-order differential equations relating the mass accelerations in three coordinate directions to the kinematic constraints of the rope. The kinematics of the model for the sliding of the mass along a rope/cable is shown in Figure 1, and the model for a 3D pendulum suitable for connection with the sliding model is shown in Figure 2.

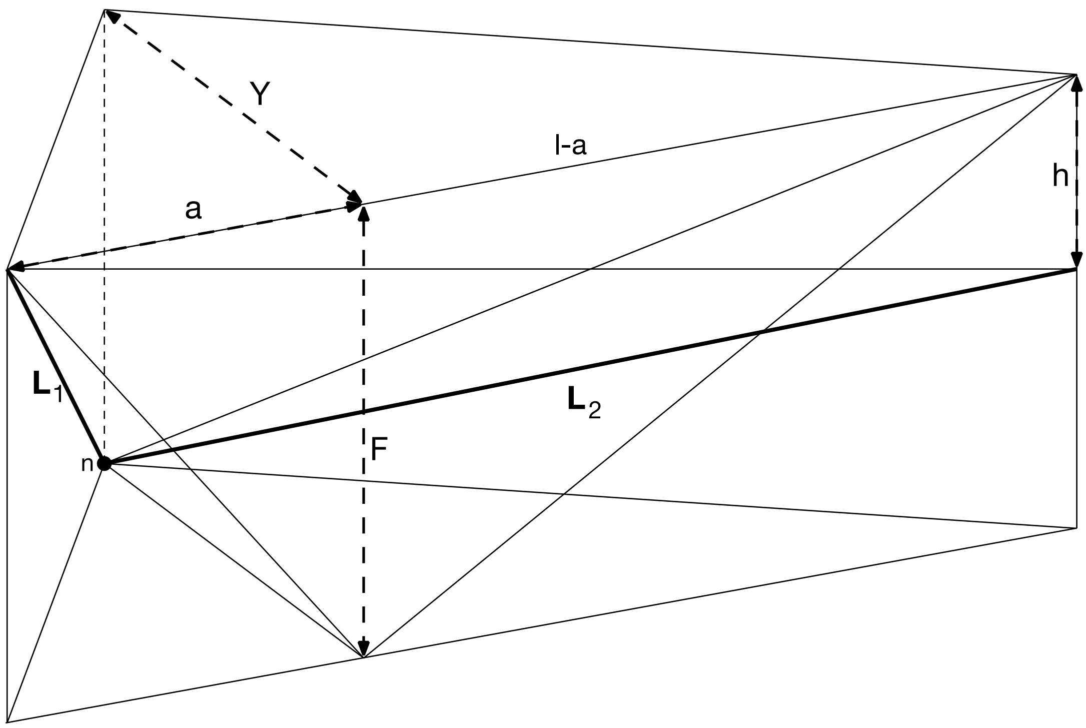

Figure 1. *Rope kinematics in 3D.*

In the case of a sliding mass, the force along the rope is constant. The differential equations describing the problem are (with the nomenclature from Fig.1)

$$\begin{Bmatrix} \ddot{a} \\ \ddot{Y} \\ \ddot{F} \end{Bmatrix} = \begin{Bmatrix} DE_1 \\ DE_2 \\ DE_3 \end{Bmatrix}, \tag{1}$$

where

$DE_1 = \frac{EA}{m} \frac{L_1+L_2}{L-1} \left(\frac{l-a}{L_2} - \frac{a}{L_1} \right)$, $\quad DE_2 = \frac{G}{m} - \frac{EA}{m} \frac{L_1+L_2}{L-1} \left(\frac{F-h}{L_2} + \frac{F}{L_1} \right)$ and

$DE_3 = -\frac{EA}{m} \frac{L_1+L_2}{L-1} \left(\frac{Y}{L_2} + \frac{Y}{L_1} \right)$.

The above equation is nonlinear because lengths 'L' are not constants but functions, i.e., $L_1 = L1(a, F, Y) = \sqrt{(a^2 + F^2 + Y^2)}$ and $L_2 = L2(a, F, Y) = \sqrt{(l-a)^2 + (F-h)^2 + Y^2}$.

2.2 The pendulum model

The pendulum model describes the 3D oscillation of a mass on an inextensible wire. The model is formulated in a local {x,y,z} coordinate system, where 'z' is not an independent coordinate because we have $z^2 = L_p^2 - x^2 - y^2$ due to the inextensible pendulum wire of length 'L_p'. The kinematics of the 3D pendulum suitable for connection to the sliding model is shown in Fig.2. Note the point 'n' in Fig.1 and Fig.2 where the two models connect.

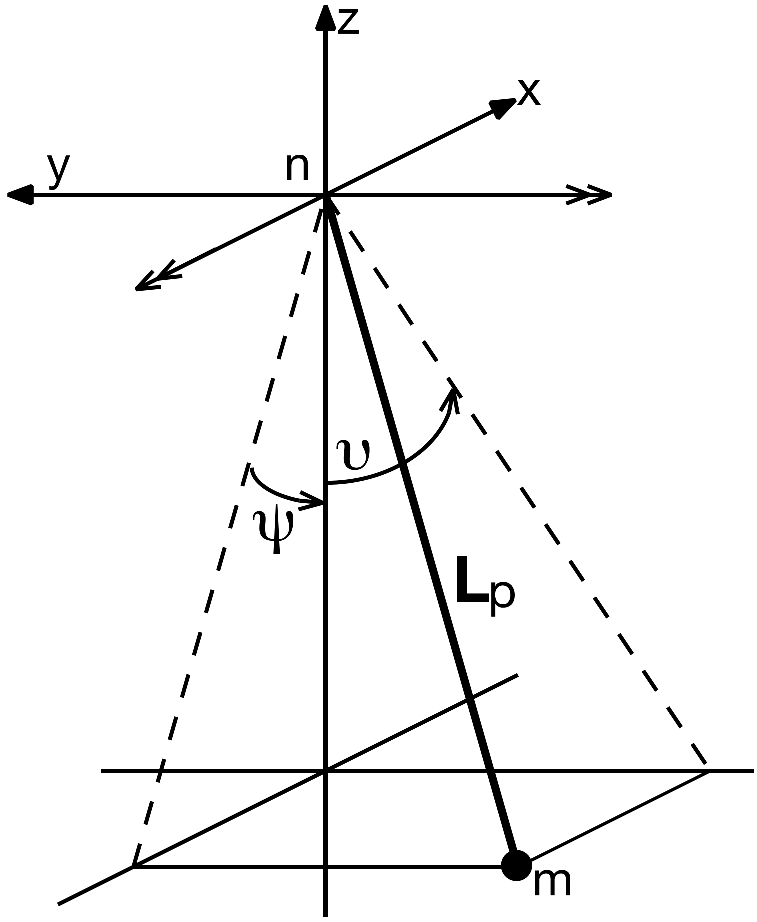

Figure 2. Pendulum kinematics in 3D.

The relationship between the relative pendulum coordinates and the pendulum angles is as follows

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \left(E_\psi F_\vartheta\right)^{-1} \begin{Bmatrix} 0 \\ 0 \\ L_p \end{Bmatrix}, \tag{2}$$

where $E_\psi = \begin{bmatrix} cos(\psi(t)) & 0 & -sin(\psi(t)) \\ 0 & 1 & 0 \\ sin(\psi(t)) & 0 & cos(\psi(t)) \end{bmatrix}$ and $F_\vartheta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\vartheta(t)) & -sin(\vartheta(t)) \\ 0 & sin(\vartheta(t)) & cos(\vartheta(t)) \end{bmatrix}$.

The accelerations of the relative pendulum coordinates are

$$\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix} = L_p \frac{d^2}{dt^2} \begin{Bmatrix} sin(\psi(t)) \\ cos(\psi(t))sin(\vartheta(t)) \\ cos(\vartheta(t))sin(\psi(t)) \end{Bmatrix}, \tag{3}$$

In the solution of the Equation 3 there is a substitute $\frac{d\vartheta}{dt} = p$ and $\frac{d\psi}{dt} = q$.

Dynamic equilibrium is described by conservation of momentum about two independent axes of rotation (along and perpendicular to the wire): ϑ and Ψ, leading to two second order differential equations (see [5] for 'classic' pendulum angles only). The equations are formulated as equilibrium of internal and external momentum, according to the d'Alembert principle

$$\vec{N_i} = \vec{N_g}, \tag{4}$$

where $\vec{N_i}$ is the internal and $\vec{N_g}$ the external momentum due to gravity. Further

$$\vec{N_i} = \left(E_\psi F_\vartheta\right)\left(\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \times m \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix}\right), \quad \vec{N_i} = \left(E_\psi F_\vartheta\right)\left(\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ -mg \end{Bmatrix}\right). \tag{5}$$

After the appropriate substitutions, we obtain four normalised differential equations $\frac{d}{dt}\begin{Bmatrix} \vartheta \\ \psi \\ p \\ q \end{Bmatrix} = \begin{Bmatrix} p \\ q \\ DEP_1 \\ DEP_2 \end{Bmatrix}$ where DEP_i are pendulum differential equations.

2.3 The combined model

The combination of the two models is based on [6]; it combines the influence of the mass sliding along an extensible wire with a 3D pendulum model. In addition, the pendulum origin moves and accelerates as it slides along the rope. We have to modify the existing equations and the Equation (1) becomes:

$$\begin{Bmatrix} \ddot{a} \\ \ddot{Y} \\ \ddot{F} \end{Bmatrix} = \begin{Bmatrix} DE_1 \\ DE_2 \\ DE_3 \end{Bmatrix} - \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix}, \tag{6}$$

where DE_i are differential equations describing changes in position along the wire. {x,y,z} are pendulum coordinates which must be related to the coordinates of the point on the wire. The momentum balance Equation (4) now gets an additional term $N_i = N_g + N_a$ that introduces the motion and acceleration of the origin of the pendulum:

$$\vec{N_a} = \left(E_\psi F_\vartheta\right)\left(\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \times \begin{Bmatrix} \ddot{a} \\ \ddot{Y} \\ \ddot{F} \end{Bmatrix}\right). \tag{7}$$

After combining the above equations and introducing the necessary transformations, our model is described by a system of 5 second order (or 10 normalized - first order) nonlinear differential equations with suitable initial conditions:

$$\begin{Bmatrix} \ddot{a} \\ \ddot{Y} \\ \ddot{F} \\ \dot{p} \\ \dot{q} \end{Bmatrix} = \begin{Bmatrix} DE_1 - \ddot{x} - fric(a) \\ DE_2 - \ddot{y} \\ DE_3 - \ddot{z} \\ DE_4 \\ DE_5 \end{Bmatrix}. \tag{8}$$

Here $DE_4 = \frac{d(\vec{N_i} - \vec{N_g} - \vec{N_a})}{d\vartheta}$, $DE_5 = \frac{d(\vec{N_i} - \vec{N_g} - \vec{N_a})}{d\psi}$ and we have introduced the frictional force, "*fric(a)*". We see that it is possible to introduce the frictional force as a function of any available parameter (here the distance 'a'). However, it is not easy to determine its functional form.

For the solution procedure, Equation (8) is expanded into the normalised system of 10 first order differential equations with 3 additional variables

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} \dot{a} \\ \dot{Y} \\ \dot{F} \end{Bmatrix}. \tag{9}$$

Here, *u*, *v*, *w* are additional variables (besides '*p*' and '*q*'). Numerical calculations are performed using Wolfram Mathematica [7] numerical solver 'NDSolve' with settings: Method->"EquationSimplification"->"Solve" and MaxSteps->100000.

2.4 The initial conditions

The initial conditions determine the dependent parameters, given the initial position of the load (pendulum). One gives the initial (starting) position on the rope ‘*a*’ and ‘*Y*’ and finds compatible displacements ‘*F*’; together they then form the initial conditions. Initially, it is determined from two nonlinear equations with unknown total sag F and wire tension force T. One equation describes the total length of the rope (including elastic elongation) and the other describes the force balance at the load position. It is possible to combine the two equations into one by inserting the length constraint into the force balance. The resulting equation is

$$mg = EA(L1 + L2 - 1)\left(\frac{F}{L1} + \frac{F-h}{L2}\right), \tag{10}$$

Here, *L1* and *L2* are functions given above. Moreover, initial condition for lateral displacement is usually zero, i.e., $Y_0 = 0$.

From the graphical representation of Equation (10) one can see the nature of the nonlinearity and the approximate solution for the initial condition. In practice, Equation (10) is solved by the secant method.

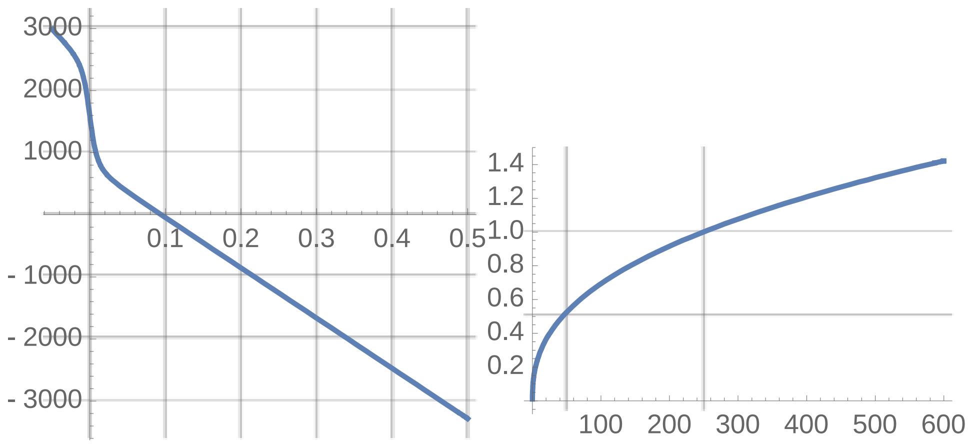

Figure 3. *(left) initial conditions, $Y_0 = 0$, a=0.01 m, (right) friction force fric(a).*

In Fig.3 we see the frictional force used to model braking on the zip-line. It can be easily introduced into the final system of equations (8), but its exact form is unknown at the moment. It has been assumed to be parabolic and a function of the distance ‘a’, with parameters adjusted to reduce the speed of sliding to approximately zero (see Fig.4). The determination of the parameters of the braking function in advance, so that it is guaranteed to stop sliding at the end of the rope, has not yet been solved.

3 EXAMPLE

This example is a dynamic recalculation of some previous designs performed with a static 2D version of the cable model used for the design of a series of zip-lines built over the river “Cetina” in Omiš, Croatia. The zip-lines varied in length, from 280 to 740 meters, and a few of them were without sag. At this point I would like to emphasise the role of the tension force in the rope/cable. Although it is deleted from the equations to reduce its number, it is important because its static value can be easily measured. The measured value can be used to estimate the geometry parameters of the rope/cable, since the length and height measurements in the field are prone to error. In Fig.4 (left) we see an example of rope/cable tensile force measurement due to self weight and static load. In Fig.4.(right) we see the tension cell connected to the data logger hanging from a person during zip-line flight and a 3-axial acceleration sensor.

Figure 4. *Tension force measurement due to static loading (left), force and acceleration measurement on the pendulum attachment point (right).*

The numerical example was calculated with Wolfram Mathematica [7] with realistic parameters from a zip-line: *EA=6 MN, m=150 kg, l=600 m, h=60 m, L=602.9 m, Lp = 0.1 m* m. The first 42 seconds of the simulation are shown in Figs. 5, 6, and 7. Fig.4 shows the change in sliding speed, the vertical position on the rope (at the point of pendulum suspension) and the tension force in the rope. This dynamic tension force can be compared with the measured static tension force (see Fig.4). Although they are different, their comparison is a good guide for design. Fig.6 shows the trajectories of the pendulum suspension point (on the rope) and the pendulum mass with displacement 'Y' exaggerated by an unknown factor (the size of the drawing frame for the image has been specified and filled in for maximum visibility). Fig.7 (left) shows an envelope of the pendulum suspension point along the sliding path and Fig.7 (right) shows the swinging of the mass on the pendulum during sliding.

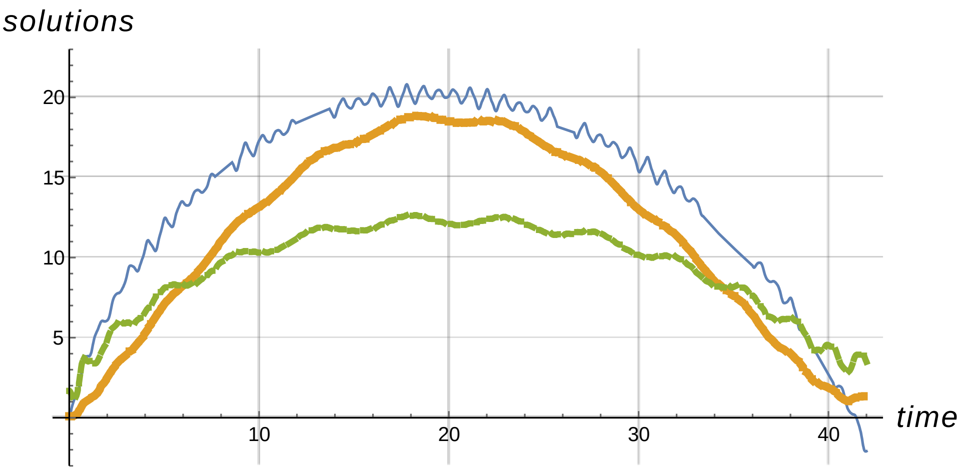

Figure 5. *Comparison of various parameters in the result.*

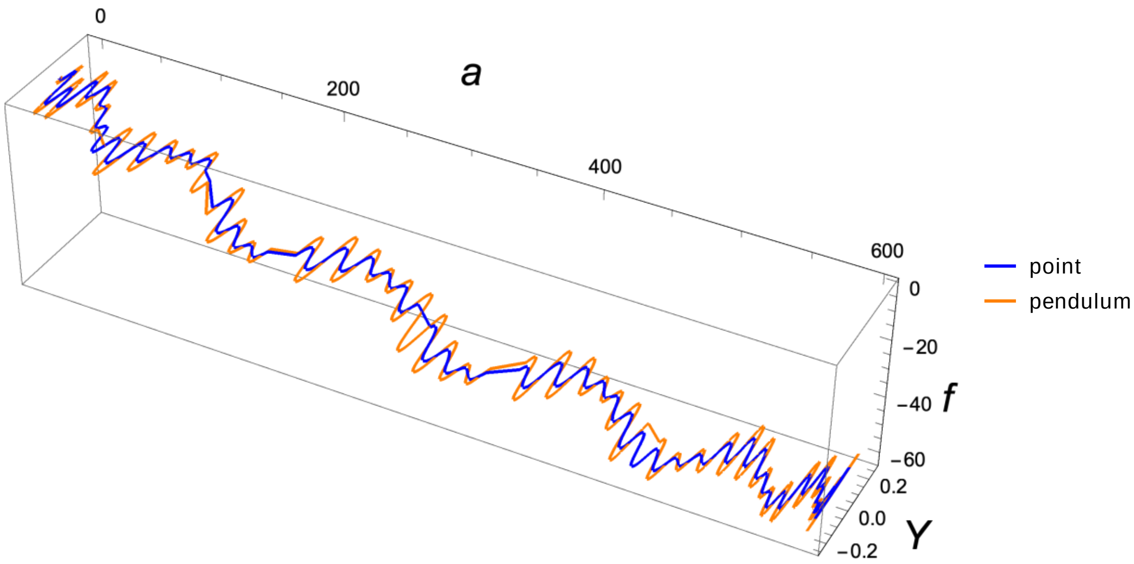

Figure 6. *Comparison of displacements of the point on the rope and the mass of the pendulum.*

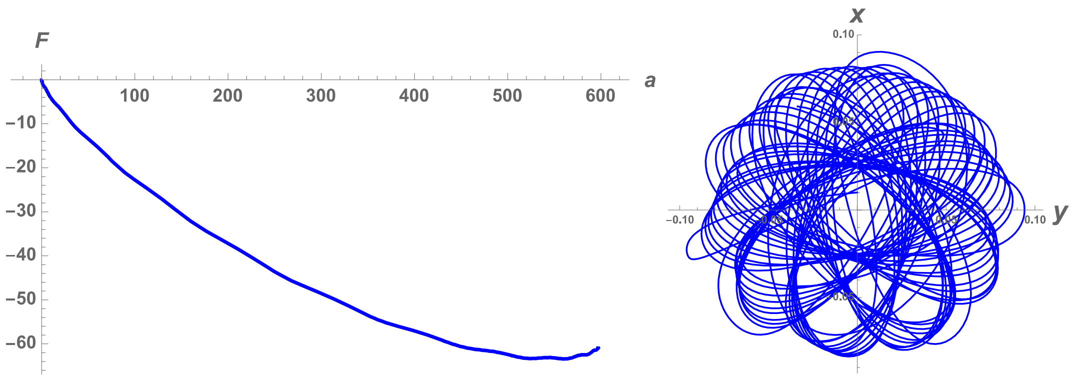

Figure 7. *(left) Side view of the rope envelopes during mass sliding, (right) top view of the pendulum mass during sliding.*

Fig.7 (left) is a lateral projection of Fig.5. Each point on this figure should be connected by a straight line to the rope/cable supports, which would obscure the figure and is therefore not included. This data could be measured with the GPS locator, but this requires a professional instrument with a sampling rate of at least 10 samples/second. From Fig.7 (right), it can be estimated that the pendulum motions are less than 10 cm. This is important data, because the accelerations of a person on the rope can be measured and the displacements can be extracted from it, so that these data can also be used for parameter calibration in the design of the zip-line.

4 CONCLUSIONS

The model based on a system of algebraic differential equations is quite successful in modeling the sliding of mass along a rope. The presented model mitigates the problem of modeling a moving load by including its position in the system variables. This model can be further discretized by any method as long as the moving load is accounted for. A numerical example illustrates the successful implementation of the model. In addition, the model has been verified on the examples of a sliding mass without pendulum and a fixed pendulum expressed by 'classical' angles and by angles υ and ψ. In addition, the model can serve as a good basis for a parametric analysis that could facilitate the zip-line design and give a better insight into the behavior of a rope under moving load. Furthermore, the model results could be related to some data that are relatively easy to measure in the field and thus could be used for parameter calibration.

In the future, brake modelling will be analysed to mimic human behaviour when stopping at the end of the rope; currently it is based on an estimate of a friction parameter. There are also plans to modify the equations to account for ropes/cables whose dead weight is not negligible (i.e., long cables and ropes with low tensile force).

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